Algebraic proof of Hodge degeneration (and others), following Deligne-Illusie

Shengxuan Liu, with Ben Gould, Jason Kountouridis, Arnab Kundu, Jennifer Li, Swaraj Pande, and Burt Totaro

September 3, 2023

Thanks to the organizers: Pieter Belmans, Johan de Jong, Wei Ho and Lena Ji.

Motivation (of myself)

In the Remark III 7.15 of Hartshorne's Algebraic Geometry published in 1977, he wrote that there is no algebraic proof of Kodaira vanishing yet. But an algebraic proof was found ten years later by Deligne-Illusie.

Remark 7.15 (The Kodaira Vanishing Theorem). Our discussion of the cohomology of projective varieties would not be complete without mentioning the Kodaira vanishing theorem. It says if X is a projective nonsingular variety of dimension n over **C**, and if $\mathscr{L}$ is an ample invertible sheaf on X, then:

(a) $H^i(X, \mathscr{L} \otimes \omega) = 0$ for $i > 0$;
(b) $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < n$.

248

---

7 The Serre Duality Theorem

Of course (a) and (b) are equivalent to each other by Serre duality. The theorem is proved using methods of complex analytic differential geometry. At present there is no purely algebraic proof. On the other hand, Raynaud has

Main Theorem

Theorem (Deligne-Illusie, 1987)

Let X be a smooth scheme over a perfect field k of characteristic $p > 0$. Each lift $\widetilde{X}$ of X to $W_2(k)$ canonically determines an isomorphism

$$\varphi_{\widetilde{X}} \colon \ \oplus_{i<p} \Omega^i_{X'/k}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/k}$$

in the derived category $D(\mathcal{O}_{X'})$. The isomorphism has the property that $\mathcal{H}^i \varphi_{\widetilde{X}} = C^{-1}$ for $i < p$.

Witt vector

The Witt vector is the unique flat lifting of Spec(k)

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(W(k)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_p). \end{array}$$

Proof

Lemma 1

To give the desired isomorphism in the main theorem in $D(\mathcal{O}_{X'})$,

$$\varphi_{\tilde{X}} \colon \oplus_{i<p} \Omega^i_{X'/S}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/S},$$

it is sufficient to give a morphism in degree 1,

$$\varphi^1_{\tilde{X}} \colon \Omega^1_{X'/S}[-1] \to \tau_{<p} F_* \Omega^\bullet_{X/S}$$

in $D(\mathcal{O}_{X'})$, such that $\mathcal{H}^1 \varphi^1_{\tilde{X}} = C^{-1}$.

Idea of proof:

1. Suppose one has a desired map in degree 1, use antisymmetrisation map to extend it to the whole complex.
2. By the multiplicative property of the Cartier isomorphism, it follows that the extended morphism has the desired property.

Proof

Lemma 2

Let X be a smooth scheme over a perfect field k of characteristic $p > 0$. Suppose that X lifts to a scheme $\widetilde{X}$ over $W_2(k)$, and suppose that the Frobenius morphism $F\colon X \to X'$ also lifts to $W_2(k)$. Then these lifts determine a morphism of complexes $f\colon \Omega^1_{X'/k}[-1] \to F_*\Omega^1_{X/k}$ such that $\mathcal{H}^1 f = C^{-1}$.

The map $F^*\colon \Omega^1_{X'/S} \to F_*\Omega^1_{X/S}$ is zero, so the image of $\widetilde{F}^*\colon \Omega^1_{\widetilde{X'}/\widetilde{S}} \to \widetilde{F}_*\Omega^1_{\widetilde{X}/\widetilde{S}}$ is contained in $p\widetilde{F}_*\Omega^1_{\widetilde{X}/\widetilde{S}}$. Then use the following diagram to construct f and show that $\mathcal{H}^1 f = C^{-1}$.

$$\begin{array}{ccc} \Omega^1_{\widetilde{X'}/\widetilde{S}} & \xrightarrow{\widetilde{F}^*} & p\widetilde{F}_*\Omega^1_{X/S} \\ \downarrow & & \cong \uparrow \mathbf{p} \\ \Omega^1_{X'/S} & \xrightarrow{f} & F_*\Omega^1_{X/S}. \end{array}$$

Proof

Lemma 3

Let k be a perfect field of characteristic $p > 0$, $S = \mathrm{Spec}(k)$, $\widetilde{S} = \mathrm{Spec}(W_2(k))$, and X a smooth scheme over k. Let $\widetilde{X}$ be a lift of X to $W_2(k)$. Let $F\colon X \to X'$ be the relative Frobenius morphism of X/k. Suppose that there are lifts $\widetilde{F}_1\colon \widetilde{X} \to \widetilde{X'}$ and $\widetilde{F}_2\colon \widetilde{X} \to \widetilde{X'}$ over $\widetilde{S}$ of the relative Frobenius morphism $F\colon X \to X'$. Then these data determine an $\mathcal{O}_{X'}$-linear map

$$h_{12}\colon \Omega^1_{X'/S} \to F_*\mathcal{O}_X$$

such that $f_2 - f_1 = \mathrm{d}h_{12}$, where $f_i = p^{-1}\widetilde{F}_i^{\,*}\colon \Omega^1_{X'/S} \to F_*\Omega^1_{X/S}$. Given three lifts of Frobenius $\widetilde{F}_i\colon \widetilde{X} \to \widetilde{X}'$ over $\widetilde{S}$, the homomorphisms $h_{ij}\colon \Omega^1_{X'/S} \to F_*\mathcal{O}_X$ associated to $\widetilde{F}_j - \widetilde{F}_i$ satisfy $h_{12} + h_{23} = h_{13}$.

This is by noticing that $\alpha := \widetilde{F_2}^* - \widetilde{F_1}^*\colon \mathcal{O}_{\widetilde{X'}} \to p\widetilde{F}_*\mathcal{O}_{\widetilde{X}} = \mathbf{p}F_*\mathcal{O}_X$ is a derivation over $\mathcal{O}_S$.

Proof

1. Locally one can always lift the relative Frobenius morphism.
2. Take an affine open cover $\mathcal{U}$. Define $\check{\mathcal{C}}(\mathcal{U}, \Omega^\bullet_{X/S})^n = \oplus_{a+b=n}\check{\mathcal{C}}^b(\mathcal{U}, \Omega^a_{X/S})$ with differential $d = d_1 + d_2$, where d_1 is the differential on the form, and d_2 is the differential associated to a Čech complex up to a sign, then this complex is quasi-isomorphic to $\Omega^\bullet_{X/S}$.
3. Define the morphism

$$\varphi^1_{(\mathcal{U},(\widetilde{F}_i))} : \Omega^1_{X'/S} \to F_*\check{\mathcal{C}}(\mathcal{U}, \Omega^\bullet_{X/S})^1 = F_*\check{\mathcal{C}}^1(\mathcal{U}, \mathcal{O}_X) \oplus F_*\check{\mathcal{C}}^0(\mathcal{U}, \Omega^1_{X/S})$$

by

$$(\varphi_1\omega)(i,j) = h_{ij}(\omega|_{U'_{ij}}), \quad (\varphi_2\omega)(i) = f_i(\omega|_{U'_i}).$$

Use the relations in the last lemma, we have $\mathrm{d}\varphi^1_{(\mathcal{U},(\widetilde{F}_i))} = 0$ and thus $\varphi^1_{(\mathcal{U},(\widetilde{F}_i))}$ extends to $F_*\check{\mathcal{C}}(\mathcal{U}, \Omega^\bullet_{X/S})^\bullet$. By the second lemma, we have $\mathcal{H}^1(F_*\check{\mathcal{C}}(\mathcal{U}, \Omega^\bullet_{X/S})) = C^{-1}$, and by the first lemma, we finish the proof.

Splitting theorem for smooth schemes of dimension $\leq p$

Lemma (Deligne–Illusie, 1987)

Let X be a smooth scheme of dimension $\leq p$ over a perfect field k of characteristic p. If X lifts to $W_2(k)$, then the complex $F_*\Omega^\bullet_{X/S}$ is isomorphic in $D(\mathcal{O}_{X'})$ to a complex with zero differential. More precisely, $F_*\Omega^\bullet_{X/S}$ is isomorphic to the complex $\oplus_i \Omega^i_{X'/S}[-i]$.

Idea of proof:

1. For degree or dimension $< p$, this is already done in the main theorem.
2. Consider the exact triangle

$$\tau_{<p} F_*\Omega^\bullet_{X/S} \to F_*\Omega^\bullet_{X/S} \to \mathcal{H}^p[-p] \underset{e}{\to} \tau_{<p} F_*\Omega^\bullet_{X/S}[1], \tag{1}$$

   reduce to calculate $e_i \in \mathrm{Hom}_{X'}(\mathcal{H}^p[-p], \mathcal{H}^i[1-i]) = H^{p-i+1}(X', \mathcal{H}om(\mathcal{H}^p, \mathcal{H}^i))$ which is 0 for $i \neq 0$ by duality and the main theorem, and for $i = 0$ by the dimension reason.

Hodge degeneration in characteristic $p > 0$

Theorem (Deligne–Illusie, 1987)

Let k be a perfect field of characteristic $p > 0$, and let X be a smooth proper k-scheme of dimension $\leq p$ that lifts to $W_2(k)$. Then the Hodge-to-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^*_{dR}(X/k)$$

degenerates at E_1.

By the spectral sequence, this reduces to a dimension count. By the splitting lemma, we have

$$\bigoplus H^{r-i}(X', \Omega^i_{X'/k}) \xrightarrow{\sim} H^r(X', F_* \Omega^\bullet_{X/k}).$$

Then it is direct as F is finite.

Kodaira–Akizuki–Nakano vanishing theorem in characteristic $p > 0$

Theorem (Deligne–Illusie, 1987)

Let k be a perfect field of characteristic $p > 0$. Let X be a smooth projective variety of dimension n over k that lifts to $W_2(k)$, and let L be an ample invertible sheaf on X. Then

$$H^j(X, \Omega^i_{X/k} \otimes L) = 0 \text{ for all } i + j > \max(n, 2n - p), \tag{2}$$

and the dual invertible sheaf $L^* = L^{\otimes -1} = \mathcal{H}om(L, \mathcal{O}_X)$ satisfies

$$H^j(X, \Omega^i_{X/k} \otimes L^*) = 0 \text{ for all } i + j < \min(n, p). \tag{3}$$

Idea of Proof

1. $H^j(X, \Omega^i_{X/k} \otimes M) = 0$ can be deduced from $H^j(X, \Omega^i_{X/k} \otimes M^{\otimes p}) = 0$ by the main theorem and the projection formula.
2. Use Serre vanishing theorem and take $M = (L^*)^{p^k}$ for k big enough.

Characteristic 0 theorems

Hodge Degeneration Theorem (Deligne–Illusie, 1987)

Let X be a smooth proper variety over a field K of characteristic zero. Then the Hodge-to-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_{X/K}) \Rightarrow H^*_{dR}(X/K)$$

degenerates at E_1.

KAN Vanishing Theorem (Deligne–Illusie, 1987)

Let X be a smooth projective variety of dimension n over a field K of characteristic zero, and let L be an ample invertible sheaf on X. Then

$$H^j(X, \Omega^i_{X/k} \otimes L) = 0 \text{ for all } i + j > n$$

and

$$H^j(X, \Omega^i_{X/k} \otimes L^*) = 0 \text{ for all } i + j < n.$$

Characteristic 0 theorems, proof

1. For such an X, one can find an integral and finitely generated $\mathbb{Z}$-sub algebra A of K and a smooth proper morphism $\mathcal{X} \to S = \mathrm{Spec}(A)$ such that $X = \mathcal{X} \otimes_A K$. By taking an open subset of S if necessary, we may assume S is smooth over $\mathbb{Z}$.
2. As $R^j f_* \Omega^i_{\mathcal{X}/S}$ and $R^m f_* \Omega^\bullet_{\mathcal{X}/S}$ are coherent, by taking an open subset of S, we can assume $R^j f_* \Omega^i_{\mathcal{X}/S}$ and $R^m f_* \Omega^\bullet_{\mathcal{X}/S}$ are locally free, and thus h^{ij} and h^n are constant over S.
3. By taking a suitable closed $s \in S$ with $k(s)$ is of characteristic $p > \dim(X)$, and by smoothness of A over $\mathbb{Z}$, we know that $\mathcal{X}_s$ is liftable over $W_2(k(s))$. And thus by the corresponding theorems for characteristic p, we get the conclusion.

Thanks to our mentor, Burt Totaro.